NICTA / ANU

COMP6460/COMP4640

Reinforcement Learning and Planning under Uncertainty

Assignment 2

Maximum marks	130
Weight	10% of final grade
Submission deadline	Wednesday, 24 September 2008, 13:00
Submission mode	On paper or email to Marcus Hutter
Questions to	Marcus Hutter (preferably after class)
Estimated time	3-5 hours per lecture week $\approx$ 5-10min per mark
Late Penalty	20% per day

In the tutorial you have the opportunity to ask for further clarifications. So I recommend that you give each question a serious trial before 17 September.

Sequential Decisions based on Algorithmic Probability

MH1 (2/130) Predicting number sequences.
What is the next number in the following sequences?
(a) 1, 2, ...
(b) 1, 2, 3, 4, ...
(c) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, ...
(d) 3, 1, 4, 1, 5, 9, 2, 6, 5, 3, ...
Can these have other continuations, other than the obvious ones?
How many numbers are enough to be sure of a continuation?

MH2 (5/130) Identification of Strings & Natural Numbers.
(i) Every countable set is isomorphic ($\cong$) to $I\!N$ (by means of a bijection). Naively interpreting a string as a binary representation of a natural number is not unique. ($00101 \cong 5 \cong 101$). Construct some bijection $\langle\cdot\rangle$ between natural numbers $I\!N_0$ and strings $\{0,1\}^*$.

Unfortunately, bijections $\langle\cdot\rangle$ are not unique when concatenated,
e.g. $5 \circ 2 \cong 10 \circ 1 = 101 = 1 \circ 01 \cong 2 \circ 4$.

(ii) Develop some prefix coding for strings (called first-order), that is an injection from strings to strings ($x \to \bar{x}$) with prefix-free co-domain and $\ell(\bar{x}) = 2\ell(x) + const$.

(iii) Develop some prefix coding $x \to x$‘ (called second-order)
with $\ell(x‘) \sim \log_2(x) + 2\log_2 \log(x)$, where $\log(\langle n\rangle) := \log(n)$.

(iv) For which x is x‘ longer/shorter than $\bar{x}$, and how much?

MH3 (4/130) Upper Bound on K.
Show that $K(x) \;\leq\; \ell(x) + 2\log_2 \ell(x) + O(1)$
and $K(n) \;\leq\; \log_2 n + 2\log_2 \log n + O(1)$,
where $K = K_U$ is the prefix Kolmogorov complexity.

MH4 (15/130) (Non)Computability of Kolmogorov complexity.
Show that Kolmogorov complexity $K : I\!N \to I\!N$ is co-enumerable, (i.e. there is an algorithm that outputs a decreasing sequence of natural numbers which converges to $K(x)$) but not finitely computable. Hint: Upper and lower bound $K(u)$, where $u := \min\{n : K(n) \geq m\}$.

MH5 (8/130) Bayes' rule.
Prove Bayes' rule using (only) the axioms of probability.

Probability axioms:
(i) $P(\emptyset) = 0 \leq P(A) \leq 1 = P(\Omega)$.
(ii) $P(A \cup B) + P(A \cap B) = P(A) + P(B)$.
(iii) $P(A|B)P(B) = P(A \cap B)$.

Bayes' rule: Let D be a possible event ($P(D) > 0$) and H_i be a set of mutually exclusive hypotheses ($H_i \cap H_j = \emptyset \forall i \neq j$ and $\cup_{i\in I} = \Omega$). $P(H_i)$ is a priori plausibility of hypothesis H_i, $P(D|H_i)$ is the likelihood of event D under hypothesis H_i. Then the posterior plausibility of hypothesis H_i is $P(H_i|D) = \frac{P(D|H_i)P(H_i)}{\sum_{i\in I} P(D|H_i)P(H_i)}$.

MH6 (10/130) Relations between Complexities.
Prove that
the prefix complexity $K(x) := \min_p\{\ell(p) : U(p) = x\}$,
the monotone complexity $Km(x) := \min_p\{\ell(p) : U(p) = x*\}$, and
Solomonoff's complexity $KM(x) := -\log_2 M(x) := -\log_2 \sum_{p:U(p)=x*} 2^{-\ell(p)}$
are ordered in the following way:

$$0 \;\leq\; K(x|\ell(x)) + O(1) \;\leq\; KM(x) \;\leq\; Km(x) \;\leq\; K(x) \;\leq\; \ell(x) + 2\log_2\ell(x) + O(1)$$

MH7 (8/130) Simple Deterministic Bound.
Sequence prediction algorithms try to predict the continuation $x_t \in \{0,1\}$ of a given sequence $x_1...x_{t-1}$. Show that $\sum_{t=1}^{\infty} |1-M(x_t|x_{<t})| \leq -\ln M(x_{1:\infty})$, where M is Solomonoff's a priori distribution. Use this to show that $M(x_t|x_{<t}) \to 1$ for computable sequences $x_{1:\infty}$. Interpret the result. Why is it interesting or important.

MH8 (20/130) Entropy Bound.
For computable μ and universal M, prove

$$D_n := \sum_{t=1}^{n} \sum_{x_{1:t}} \mu(x_{1:t}) \ln \frac{\mu(x_t|x_{<t})}{M(x_t|x_{<t})} \leq K(\mu) \ln 2 + O(1)$$

Hint: Use the mixture representation of M and the telescoping property of the KL-divergence.

MH9 (8/130) Predictive Convergence of M.
Prove Solomonoff's (1978) key result

$$\sum_{t=1}^{\infty} \sum_{x_{<t}\in\{0,1\}^{t-1}} \mu(x_{<t})\Big(M(0|x_{<t}) - \mu(0|x_{<t})\Big)^2 \;\leq\; \tfrac{1}{2}\ln 2\cdot K(\mu) + O(1) \;<\; \infty$$

Now assume sequence $x_{1:\infty}$ is sampled from the *unknown* distribution μ,
i.e. the *true objective probability* of $x_{1:n}$ *is* $\mu(x_{1:n})$.
The probability of x_t given $x_{<t}$ hence is $\mu(x_t|x_{<t}) := \mu(x_{1:t})/\mu(x_{<t})$.
Show that Solomonoff's central result implies that $M(x_t|x_{<t})$ converges for $t \to \infty$ to $\mu(x_t|x_{<t})$ with probability 1.
Hint: Use the previous exercise and $2(z-y)^2 \leq y \ln \frac{y}{z} + (1-y) \ln \frac{1-y}{1-z}$
for $0 < z < 1$ and $0 \leq y \leq 1$ without prove.

MH10 (15/130) Loss Bounds.
Let $\text{Loss}(x_t, y_t) \in [0,1]$ be the received loss when taking prediction/decision/action $y_t \in \mathcal{Y}$ and $x_t \in \mathcal{X}$ is the t^{th} symbol of the sequence, revealed after the prediction.

The goal is to minimize the μ-expected loss. More generally, define the Λ_ρ *prediction scheme*, which minimizes the ρ-expected loss:

$$y_t^{\Lambda_\rho} \;:=\; \arg\min_{y_t\in\mathcal{Y}} \sum_{x_t} \rho(x_t|x_{<t})\mathrm{Loss}(x_t, y_t)$$

The μ-expected loss when some predictor Λ_ρ predicts the t^{th} symbol is

$$\mathrm{Loss}_t(\Lambda_\rho)(x_{<t}) := \sum_{x_t} \mu(x_t|x_{<t})\mathrm{Loss}(x_t, y_t^{\Lambda_\rho})$$

$\mathrm{Loss}_t(\Lambda_{\mu/\xi})$ is the loss made by the informed/universal scheme $\Lambda_{\mu/\xi}$.
Show that $\mathrm{Loss}_t(\Lambda_\mu) \leq \mathrm{Loss}_t(\Lambda)\ \forall t, \Lambda$.
Show that

$$0 \leq \mathrm{Loss}_t(\Lambda_\xi) - \mathrm{Loss}_t(\Lambda_\mu) \leq \sum_{x_t} |\xi(x_t|x_{<t}) - \mu(x_t|x_{<t})|$$

What can you conclude from this?

MH11 (15/130) Blum's and Levin's Speed Theorems.
What does Blum's speedup theorem say?
What is Levin search? How and why does it work?
What is the asymptotically fastest algorithm for all well-defined problems?
How and why does it work?
Why don't the latter two algorithms contradict Blum's result?

MH12 (20/130) CompLearn Toolkit.
Reproduce the phylogenetic tree of mammals and the language tree using the CompLearn Toolkit available from http://www.complearn.org/