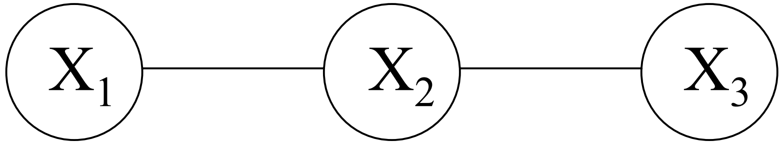
Today

- The meaning of the "potential functions" Ψ
- Markov properties
- The Elimination Algorithm
- The Junction Tree Algorithm

Meaning of potential functions

What are the "potential functions" ? Ψ



$$p(x_1x_2x_3) = p(x_1 \mid x_2x_3)\,p(x_2x_3)$$

$$p(x_1x_2x_3) = p(x_1 \mid x_2)\,p(x_2 \mid x_3)\,p(x_3)$$

$$p(x_1x_2x_3) = \frac{p(x_1x_2)}{p(x_2)}\,\frac{p(x_2x_3)}{p(x_3)}\,p(x_3)$$

$$p(x_1x_2x_3) = \frac{p(x_1x_2)\,p(x_2x_3)}{p(x_2)}$$

$$p(x_1x_2x_3) = \Psi_{12}(x_1x_2)\Psi_{23}(x_2x_3)$$

Where

$$\Psi_{12}(x_1x_2) = \frac{p(x_1x_2)}{p(x_2)}$$

$$\Psi_{23}(x_2x_3) = p(x_2x_3)$$

Meaning of potential functions

Think of the potential function $\Psi_c(x_c)$ as the "likelihood" of the assignment x_c

(since they are not necessarily probabilities)

Potential functions

Example:

The potential function of a pair of neighbouring pixels can be HIGH if the pixels have the same label or SMALL if they have different labels

$$\Psi_{1,2}(x_1, x_2)$$

$x_1 \backslash x_2$	0	1
0	10	2
1	2	10

$$\Psi_{1,2}(0,0) = 10$$
$$\Psi_{1,2}(0,1) = 2$$
$$\Psi_{1,2}(1,0) = 2$$
$$\Psi_{1,2}(1,1) = 10$$

Markov properties

Pairwise Markov property:

$$X_i \perp X_j \mid V \setminus \{X_i, X_j\}$$

Local Markov property:

$$X_i \perp (V \setminus \{X_i \bigcup bd(X_i)\} \mid bd(X_i)$$

Global Markov property:

$$X_A \perp X_B \mid X_S$$

Elimination

Now we have smaller tables

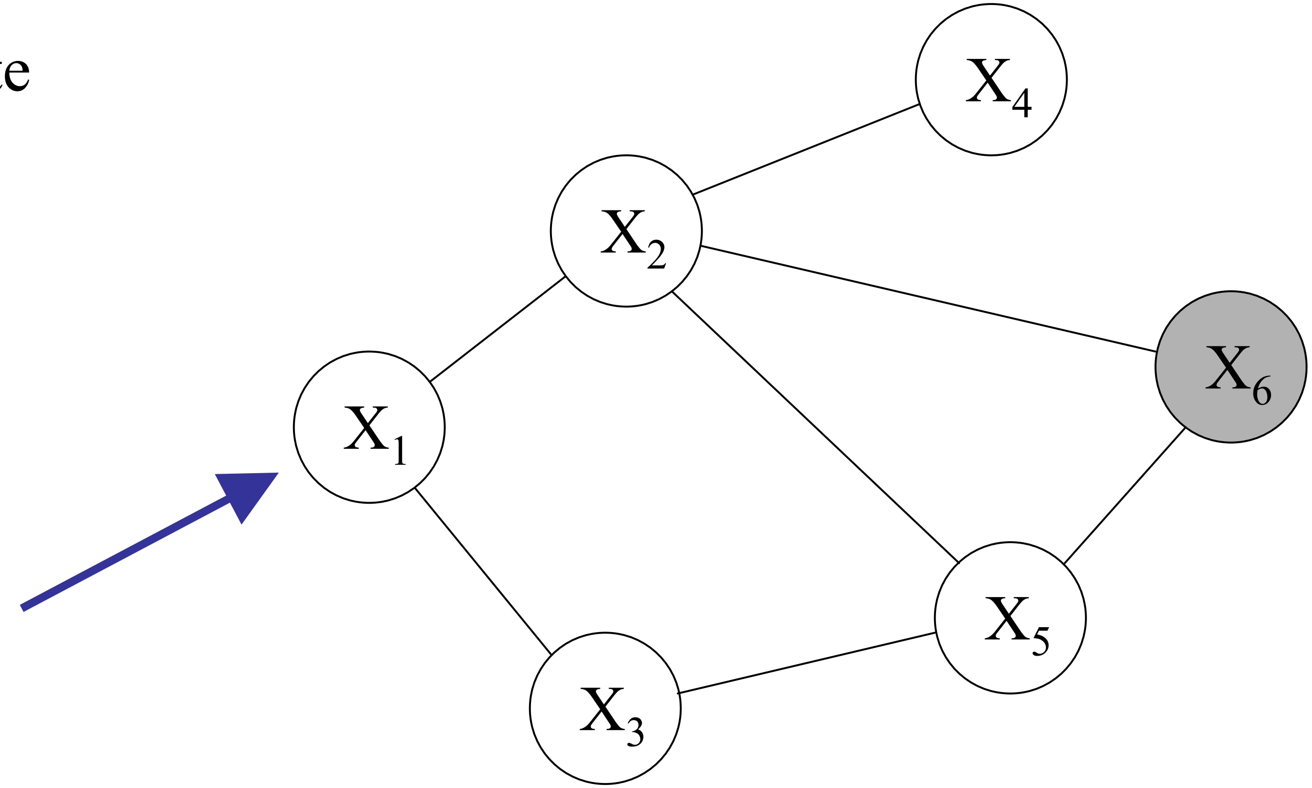
How to take advantage of the Markov structure of the graph to perform inference?

Elimination algorithm

Elimination: example

How to compute

$$p(x_1 \mid \bar{x}_6)$$



Elimination

$$p(x_1,\bar{x}_6) = \frac{1}{Z}\sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5}\sum_{x_6}\psi(x_1,x_2)\psi(x_1,x_3)\psi(x_2,x_4)\psi(x_3,x_5)\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

$$p(x_1,\bar{x}_6) = \frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)\sum_{x_3}\psi(x_1,x_3)\sum_{x_4}\psi(x_2,x_4)\sum_{x_5}\psi(x_3,x_5)\sum_{x_6}\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

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$$p(x_1,\bar{x}_6) = \frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)\sum_{x_3}\psi(x_1,x_3)m_5(x_2,x_3)\sum_{x_4}\psi(x_2,x_4)$$

$$p(x_1,\bar{x}_6) = \frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)m_4(x_2)\sum_{x_3}\psi(x_1,x_3)m_5(x_2,x_3)$$

$$p(x_1,\bar{x}_6) = \frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)m_4(x_2)m_3(x_1,x_2)$$

$$p(x_1,\bar{x}_6) = \frac{1}{Z}m_2(x_1)$$

Elimination

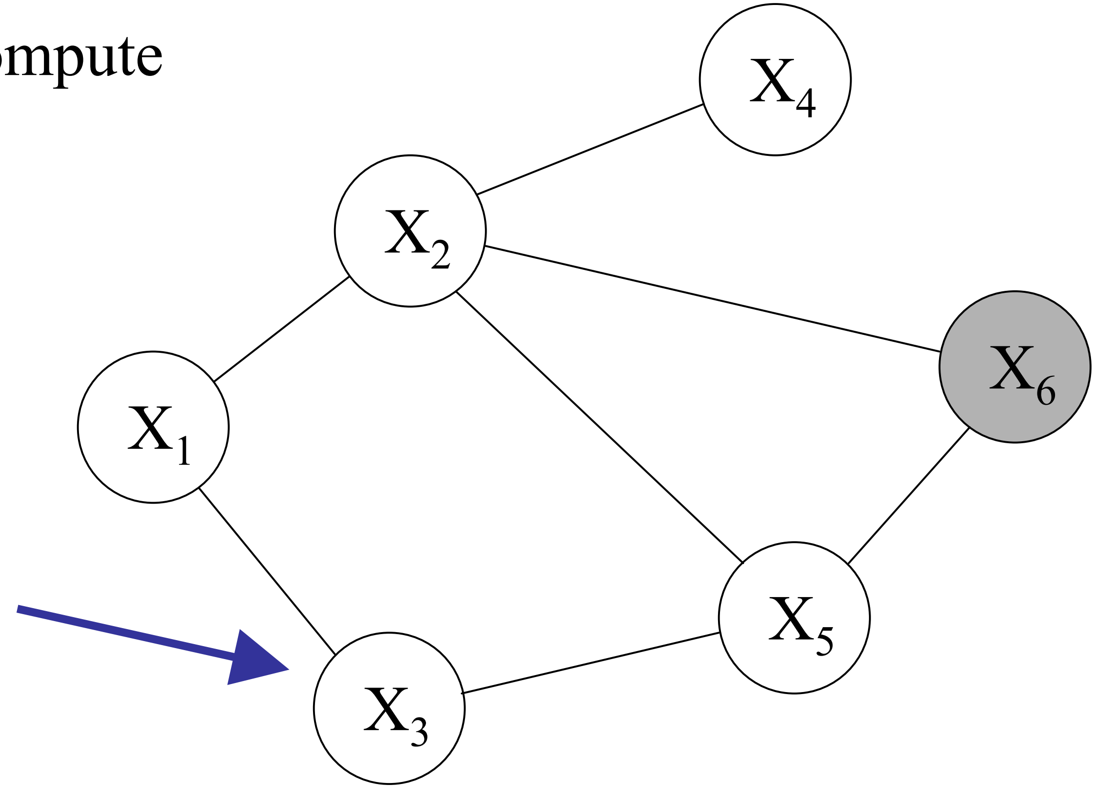
$$p(\bar{x}_6) = \frac{1}{Z}\sum_{x_1} m_2(x_1) \qquad p(x_1, \bar{x}_6) = \frac{1}{Z} m_2(x_1)$$

$$p(x_1 \mid \bar{x}_6) = \frac{m_2(x_1)}{\sum_{x_1} m_2(x_1)}$$

Elimination: example

What if now we want to compute

$$p(x_3 \mid \bar{x}_6)$$



Elimination: example

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\sum_{x_2}\sum_{x_4}\sum_{x_5}\sum_{x_6}\psi(x_1,x_2)\psi(x_1,x_3)\psi(x_2,x_4)\psi(x_3,x_5)\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)\sum_{x_4}\psi(x_2,x_4)\sum_{x_5}\psi(x_3,x_5)\sum_{x_6}\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)\sum_{x_4}\psi(x_2,x_4)\sum_{x_5}\psi(x_3,x_5)m_6(x_2,x_5)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)m_5(x_2,x_3)\sum_{x_4}\psi(x_2,x_4)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)m_4(x_2)m_5(x_2,x_3)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}m_2(x_1,x_3)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}m_1(x_3)$$

Elimination: example

Repeated Computations!!

$$p(x_1 \mid \bar{x}_6)$$

$$p(x_3 \mid \bar{x}_6)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\sum_{x_2}\sum_{x_4}\sum_{x_5}\sum_{x_6}\psi(x_1,x_2)\psi(x_1,x_3)\psi(x_2,x_4)\psi(x_3,x_5)\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)\sum_{x_4}\psi(x_2,x_4)\sum_{x_5}\psi(x_3,x_5)\sum_{x_6}\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)\sum_{x_4}\psi(x_2,x_4)\sum_{x_5}\psi(x_3,x_5)m_6(x_2,x_5)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)m_5(x_2,x_3)\sum_{x_4}\psi(x_2,x_4)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}\psi(x_1,x_3)\sum_{x_2}\psi(x_1,x_2)m_4(x_2)m_5(x_2,x_3)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}\sum_{x_1}m_2(x_1,x_3)$$

$$p(x_3,\bar{x}_6)=\frac{1}{Z}m_1(x_3)$$

$$p(x_1,\bar{x}_6)=\frac{1}{Z}\sum_{x_2}\sum_{x_3}\sum_{x_4}\sum_{x_5}\sum_{x_6}\psi(x_1,x_2)\psi(x_1,x_3)\psi(x_2,x_4)\psi(x_3,x_5)\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

$$p(x_1,\bar{x}_6)=\frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)\sum_{x_3}\psi(x_1,x_3)\sum_{x_4}\psi(x_2,x_4)\sum_{x_5}\psi(x_3,x_5)\sum_{x_6}\psi(x_2,x_5,x_6)\delta(x_6,\bar{x}_6)$$

$$p(x_1,\bar{x}_6)=\frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)\sum_{x_3}\psi(x_1,x_3)\sum_{x_4}\psi(x_2,x_4)\sum_{x_5}\psi(x_3,x_5)m_6(x_2,x_5)$$

$$p(x_1,\bar{x}_6)=\frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)\sum_{x_3}\psi(x_1,x_3)m_5(x_2,x_3)\sum_{x_4}\psi(x_2,x_4)$$

$$p(x_1,\bar{x}_6)=\frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)m_4(x_2)\sum_{x_3}\psi(x_1,x_3)m_5(x_2,x_3)$$

$$p(x_1,\bar{x}_6)=\frac{1}{Z}\sum_{x_2}\psi(x_1,x_2)m_4(x_2)m_3(x_1,x_2)$$

$$p(x_1,\bar{x}_6)=\frac{1}{Z}m_2(x_1)$$

How to avoid that?

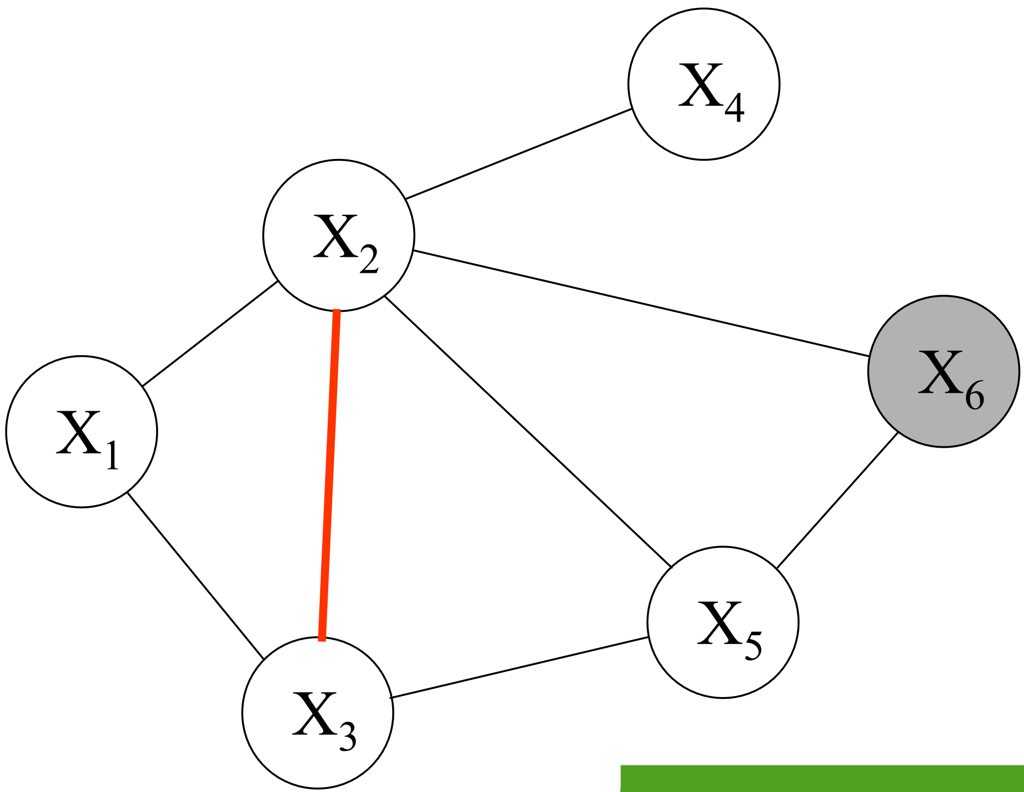
The Junction Tree Algorithm

The Junction Tree algorithm provides the definitive answer on how to perform exact inference in Graphical Models

It does not repeat computations

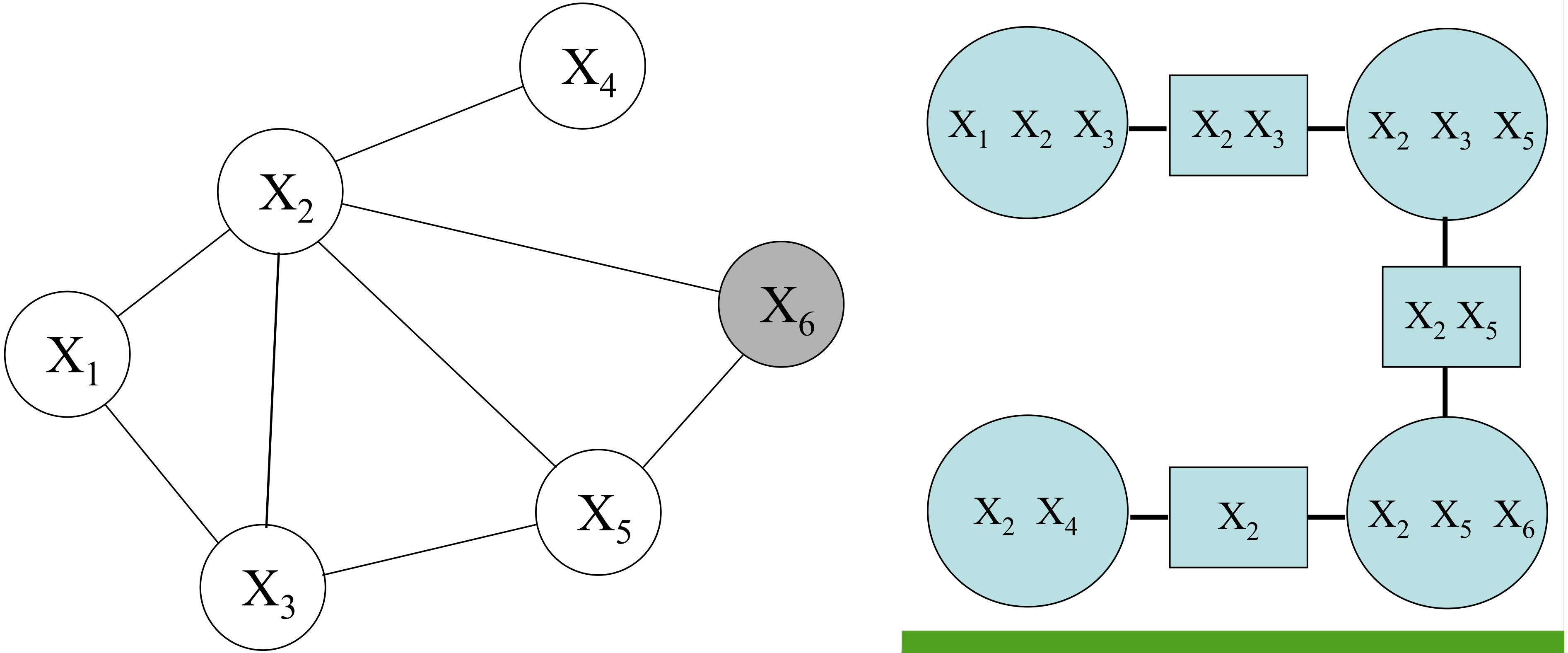
The Junction Tree Algorithm

(1) *Triangulate* the graph (if it's not triangulated)



The Junction Tree Algorithm

(2) Create a Junction Tree

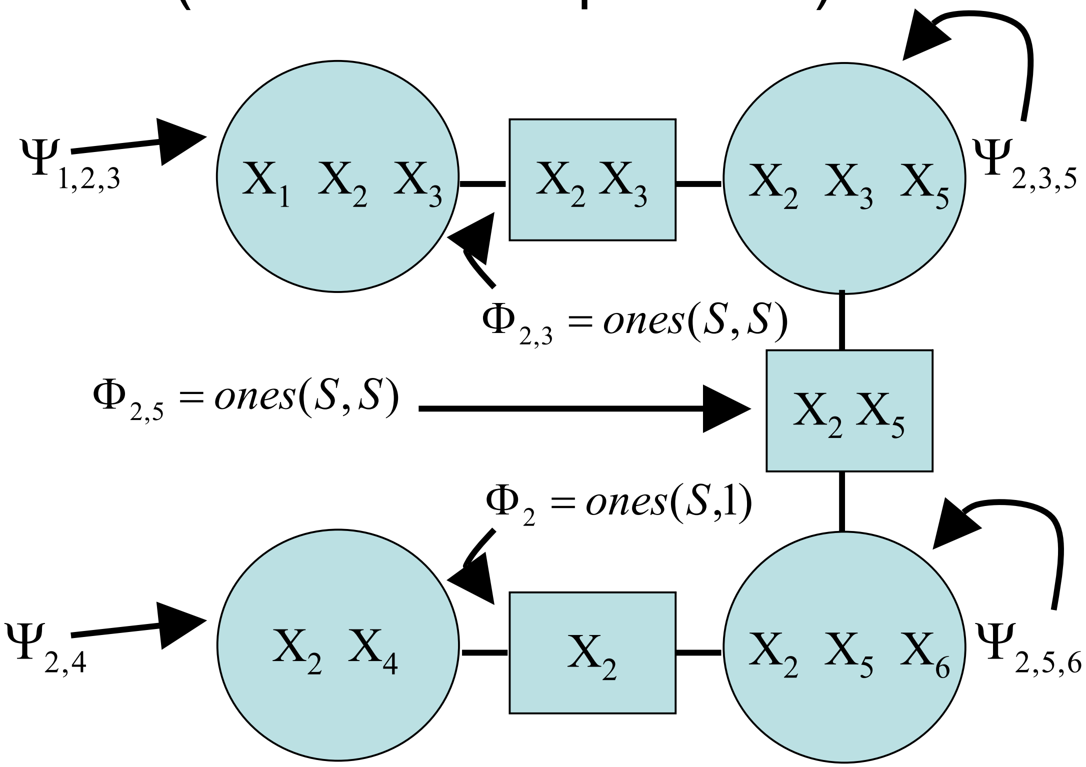


The Junction Tree Algorithm

(3) Initialize clique potentials (nodes and separators)

Ψ_c Directly introduced

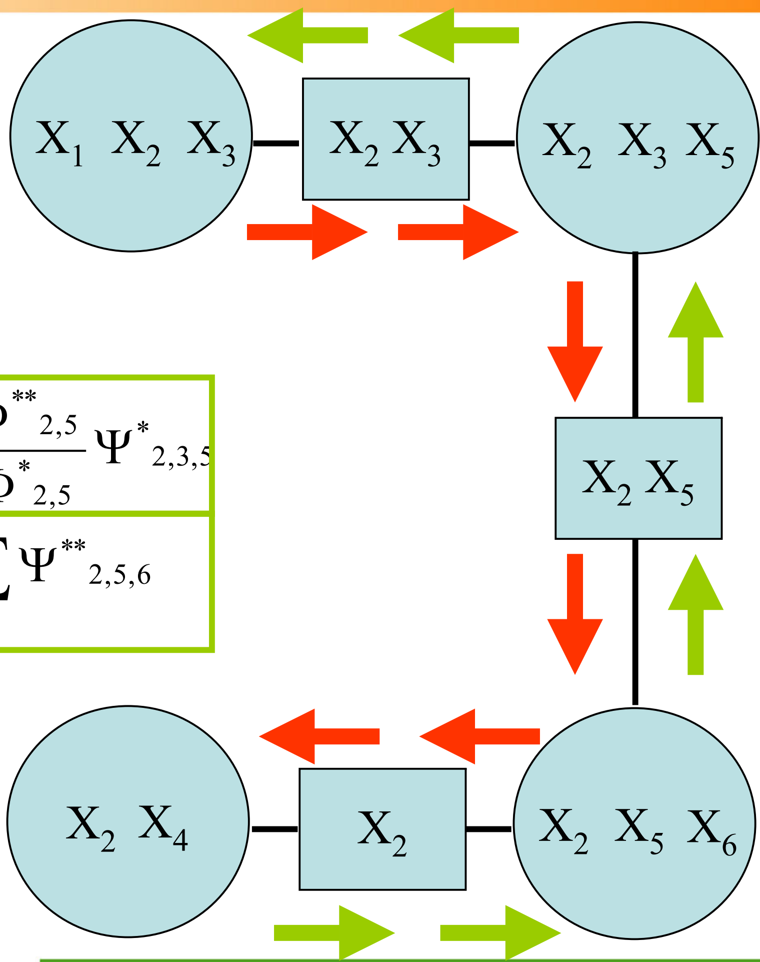
Φ_s Initialized to 1



The Junction Tree Algorithm

(4) Message passing

$\Psi^{**}_{1,2,3} = \frac{\Phi^{**}_{2,3}}{\Phi^{*}_{2,3}} \Psi_{1,2,3}$	$\Phi^{**}_{2,3} = \sum_{x_5} \Psi^{**}_{2,3,5}$	
$\Phi^{*}_{2,3} = \sum_{x_1} \Psi_{1,2,3}$	$\Psi^{*}_{2,3,5} = \frac{\Phi^{*}_{2,3}}{\Phi_{2,3}} \Psi_{2,3,5}$	
	$\Phi^{*}_{2,5} = \sum_{x_3} \Psi^{*}_{2,3,5}$	$\Psi^{**}_{2,3,5} = \frac{\Phi^{**}_{2,5}}{\Phi^{*}_{2,5}} \Psi^{*}_{2,3,5}$
	$\Psi^{*}_{2,5,6} = \frac{\Phi^{*}_{2,5}}{\Phi_{2,5}} \Psi_{2,5,6}$	$\Phi^{**}_{2,5} = \sum_{x_6} \Psi^{**}_{2,5,6}$
$\Psi^{*}_{2,4} = \frac{\Phi^{*}_{2}}{\Phi_{2}} \Psi_{2,4}$	$\Phi^{*}_{2} = \sum_{x_5 x_6} \Psi^{*}_{2,5,6}$	
$\Phi^{**}_{2} = \sum_{x_4} \Psi^{*}_{2,4}$	$\Psi^{*}_{2,5,6} = \frac{\Phi^{**}_{2}}{\Phi^{*}_{2}} \Psi_{2,5,6}$	



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The Junction Tree Algorithm

Once the algorithm has finished:

potential in each clique node is equal to the marginal in that node

Marginals for singletons can then be computed by brute force